Indian Statistical Institute
Second Semester Examination 2004-2005
M.Math II Year
Operator Theory

Time: 3 hrs Date: 13-05-2005

Answer all questions. They carry equal marks.

1. (a) Let T be a linear operator on a Hilbert space that maps every strongly convergent sequence into a weakly convergent sequence. Prove that T is bounded.

   (b) Find the polar decomposition of the operator M of multiplication by a function $\varphi \in L^\infty(X,\mu)$ on $L^2(X,\mu)$, where (X,μ) is a measure space.

2. An operator M consists of multiplication by the function $\varphi(x) = x$ on $L^2[0,1]$, and $T = M \oplus M$ on $\mathcal{H} = L^2[0,1] \oplus L^2[0,1]$.

   (a) Prove that T has no cyclic vector.

   (b) There exists a subspace $\mathcal{M}$ of $\mathcal{H}$ such that

   i) $\mathcal{M}$ is reducing for T,

   ii) the map $A \to A_{|\mathcal{M}}$ defines an isometric isomorphism of the algebra $\mathcal{W}_T^*$ to $\mathcal{W}_{T_{|\mathcal{M}}}^*$, where $\mathcal{W}_T^*$ is the weak $*$ algebra generated by T, and

   iii) $T_{|\mathcal{M}}$ has a cyclic vector.

3. (a) What is the spectrum of the multiplication operator M_φ acting on $L^2[0,1]$ for φ continuous on $[0,1]$?

   (b) Prove that the spectrum of a selfadjoint A consists of precisely those points at $I\!R$ such that $\lambda(a-\epsilon,\ a+\epsilon) \neq 0$ for any $\epsilon > 0$, where λ is the spectral measure of A.

4. Derive the finite dimensional spectral theorem from the continuous functional calculus.

5. (a) What is the spectrum of the forward shift?

   (b) What is the essential spectrum of the forward shift?

6. (a) Show that if T is positive then

$$||T|| = \sup\{\langle T_x, x\rangle :\ ||x|| = 1\}$$

   (b) Compute the operator norm of a linear transformation $T : \mathbb{C}^2 \to \mathbb{C}^2$.